Journal of Fluids and Structures 19 (2004) 209–220

JOURNAL OF FLUIDS AND STRUCTURES

www.elsevier.nl/locate/jnlabr/yjfls

Hydroelastic vibrations in a two-dimensional rectangular container filled with frictionless liquid and a partly elastically covered free surface

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Received 16 October 2002; accepted 3 November 2003

Abstract

For a structure partially covering the free liquid surface of a two-dimensional rectangular container of infinite width, length l and a filling grade of height h hydroelastic vibrations have been treated for a frictionless liquid. It was found that with a partial covering of the free surface the lower coupled frequencies increase with structural area, yielding in addition a calmed down motion of the liquid. It also was found that the coupled frequencies stay always below those of the case of the rigid surface covering.

1. Introduction

With the increasing size of airplanes and their larger amount of propellants and larger propellant container dimensions, the effect of propellant sloshing upon the performance, control and stability of such vehicles is becoming more pronounced and may even be dangerous for the projected missions. These sloshing effects have led to difficulties and peculiar performance behaviors. One of the main problems appearing in such a case is the closeness of the control or autopilot frequency to the fundamental propellant frequency. Besides the maneuvering of the vehicle and wind gust inputs, a continuous excitation of the propellant oscillations is present. This is especially of importance for low aspect ratio containers, in which nearly all the liquid participates in the sloshing motion. Similar problems appear also in liquid cargo ships, road tankers and possibly space stations. This lead to the noncontrollability and even destruction of the total vehicle. Another application would be the interaction of a floating pier or an ice plate, reaching partially into the liquid.

The problem of liquid oscillations in rectangular containers has been treated previously and may be found in Lorell (1951), Graham and Rodriguez (1952), Bauer (1966a,b) and Bauer and Villanueva (1967). It was found that the subdivision of the container by vertical walls increases the natural frequencies and decreases the magnitude of the sloshing masses participating in the oscillation (Bauer and Villanueva, 1967). The disadvantage of this method, however, is the required additional weight and as a consequence of this a reduction of the payload. In addition, such a subdivision would yield two natural frequencies which usually would be lower than that of the case of a surface cover and would also exhibit larger slosh masses participating in the liquid motion. Both effects are of disadvantage to the system, leading us to prefer the method of a free liquid surface cover by a structural element. To avoid this penalty the method of covering the free liquid surface with structural elements has been investigated by Bauer and Eidel (2000) and it could be found that not only the natural frequencies of the liquid could be increased, but that the motion of the liquid

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0889-9746/$ - see front matter © 2003 Published by Elsevier Ltd.
doi:10.1016/j.jfluidstructs.2003.11.002

Nomenclature

Bo	Bond number ($\mathrm{Bo} = \varrho_L g l^2/\sigma$)
d	thickness of the beam
E	Young's modulus of elasticity
g	gravity constant ($g* = gl^3\mu/EI$)
h	liquid height
I	area moment of inertia (EI stiffness of beam)
l	width of rectangular container
t	time
$x,\ y,\ z$	Cartesian coordinates
x_0	excitation amplitude
α	coverage ratio (αl length of surface coverage (beam))
β^{*4}	$= (\omega^2\mu - \varrho_L g)l^4/EI = \omega^{*2} - g^*/\mu^*$
$\zeta(x,t)$	free surface displacement
$\bar{\zeta}(x,t)$	beam deflection
θ_0	pitching amplitude
ϱ_C	density of beam
ϱ_L	liquid density
μ	mass per unit length of beam ($\mu^* = \mu/\varrho_C l,\ \ \mu = \varrho_C d$)
σ	liquid surface tension ($\sigma^* = \sigma l^2/EI,\ \ \bar{\sigma} = \sigma/\varrho_L l^3$)
Φ	velocity potential
ω	natural frequency $\left(\omega^* = \omega l^2\sqrt{\mu/EI}\right)$
Ω	forced frequency

in the container could be calmed down considerably thus creating a larger reduction of liquid forces and moments upon the vehicle. This is especially important for low aspect ratio containers which exhibit violent liquid motions and large sloshing masses. To save as much structural weight as possible with these free liquid surface obstructions the obstructing structural element is chosen to be thin and exhibits therefore flexibility. For such a system, we investigate in the following the coupled natural frequencies and the coupled response of the liquid. For this reason we cover the free liquid surface partially with an elastic plate (beam), which motion influences the motion of the liquid in a desirable way. This interaction of structure and liquid will depend on the property of the liquid and the elastic property of the structure.

Since we are interested mainly in the lower coupled liquid frequency, we shall numerically evaluate results only in the lower frequency range. To widen the scope of our investigation we have included surface tension to cover with the following analysis also cases where the gravity constant exhibits decreased values, i.e., cases for which the Bond number Bo approaches zero. In such cases the liquid surface tension will play the dominant role for the motion of the liquid. It is common knowledge that the natural frequency of a liquid in a container increase with the decrease of the free surface area. In Bauer and Eidel (2000) the magnitude of such a change was found for a partial obstruction of the free liquid surface by a rigid structure. The following investigation presents the magnitude of the lower coupled frequency if the partial obstruction is for reasons of weight savings considered elastic and exhibits various lengths.

The method used here has been employed successfully for various mechanical systems, such as the sloshing of a viscous liquid in a circular cylindrical container (Bauer and Eidel, 1999), the thermocapillary (Bauer and Eidel, 2002) or oscillatory behavior (Bauer and Chiba, 2003) of spherical captured liquid drops, where useful results could be obtained with reasonable numerical efforts.

2. Basic equations

A rectangular container of infinite width and length l is filled with an incompressible and frictionless liquid to a height h. Its free surface is partly covered by an elastic plate. The container walls $x = 0$ and l and the bottom of the container at $z = -h$ are considered as solid walls. A part of the liquid surface at $z = 0$ (Fig. 1) is covered by an elastic plate which may have various attachments to the walls $x = 0$ and/or $x = l$. These may either be clamped, simply supported or guided, while the other end of the elastic plate (beam) is treated as a free boundary. In the present treatment we shall

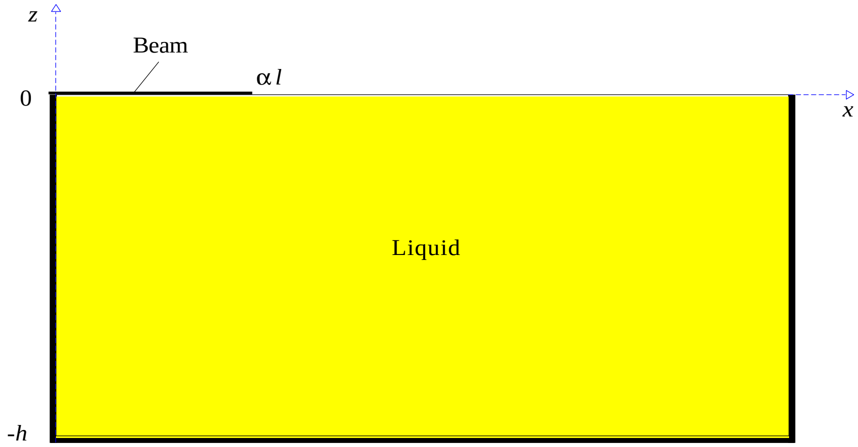

Fig. 1. Geometry and coordinate system.

restrict ourselves to an elastic cover, being clamped at the sidewall $x = 0$. Assuming the liquid to be in irrotational motion (**curl** $\vec{v} = 0$), the velocity distribution may be represented as a gradient of a velocity potential $\Phi(x, z, t)$, i.e., $\vec{v} =$ **grad** Φ. The continuity equation **div** $\vec{v} = 0$ yields then the Laplace equation

$$\Delta\Phi = 0, \tag{1}$$

which has to be solved with the rigid wall conditions

$$\frac{\partial \Phi}{\partial x} = 0 \quad \text{at the sidewalls } x = 0 \quad \text{and} \quad x = l \tag{2}$$

and

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at the container bottom } z = -h. \tag{3}$$

The free liquid surface at $z = 0$ is obtained from the kinematic condition $\partial\zeta/\partial t = \partial\Phi/\partial z$ and the dynamic condition $\partial\Phi/\partial t + g\zeta - (\sigma/\varrho_L)(\partial^2\zeta/\partial x^2) = 0$, yielding in its combined form the expression

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} - \frac{\sigma}{\varrho_L}\frac{\partial^3 \Phi}{\partial x^2 \partial z} = 0 \quad \text{at } z = 0 \text{ in the range } \alpha l < x \leqslant l, \ (0 \leqslant \alpha < 1) \tag{4}$$

and is valid in the range of the free liquid surface. In this equation ϱ_L is the density of the liquid, σ its surface tension, g is the gravity constant, while $\zeta(x, t)$ is the free surface elevation. The motion of the elastic beam at the surface $z = 0$ is given by the beam equation

$$EI\frac{\partial^4 \bar{\zeta}}{\partial x^4} + \mu\frac{\partial^2 \bar{\zeta}}{\partial t^2} = -\varrho_L\frac{\partial \Phi}{\partial t} - \varrho_L g\bar{\zeta} \quad \text{at } z = 0 \tag{5}$$

and the compatibility condition

$$\frac{\partial \bar{\zeta}}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = 0 \text{ in the range } 0 \leqslant x \leqslant \alpha l. \tag{6}$$

In Eq. (5) EI is the stiffness of the beam, μ its mass/unit length and $\bar{\zeta}(x, t)$ is the deflection of the beam. If there is more than one beam, say another at $x = l$ we have to observe additional elastic equations (5), (6). The boundary conditions for the beam are given by

$$\bar{\zeta} = \frac{\partial \bar{\zeta}}{\partial x} = 0 \quad \text{(clamped)} \quad \text{at } x = 0 \tag{7}$$

and

$$\frac{\partial^2 \bar{\zeta}}{\partial x^2} = \frac{\partial^3 \bar{\zeta}}{\partial x^3} = 0 \quad \text{(free end)} \quad \text{at } x = \alpha l. \tag{8}$$

($\alpha<1$) for the clamped–free beam,

$$\bar{\zeta}=\frac{\partial^2\bar{\zeta}}{\partial x^2}\quad \text{at } x=0 \quad \text{and} \quad \frac{\partial^2\bar{\zeta}}{\partial x^2}=\frac{\partial^3\bar{\zeta}}{\partial x^3}=0 \quad \text{at } x=\alpha l \tag{9}$$

for the simply supported—free beam, and

$$\frac{\partial\bar{\zeta}}{\partial x}=\frac{\partial^3\bar{\zeta}}{\partial x^3}=0 \quad \text{at } x=0 \quad \text{and} \quad \frac{\partial^2\bar{\zeta}}{\partial x^2}=\frac{\partial^3\bar{\zeta}}{\partial x^3}=0 \quad \text{at } x=\alpha l \tag{10}$$

for the guided-free beam. If the attachment to the container wall is elastically supported, the boundary conditions would read there

$$EI\,\frac{\partial^2\bar{\zeta}}{\partial x^2}=k\,\frac{\partial\bar{\zeta}}{\partial x}, \quad EI\,\frac{\partial^3\bar{\zeta}}{\partial x^3}=-K\bar{\zeta} \quad \text{at } x=0, \tag{11}$$

where the edge rotation is opposed by the torsional spring having the distributed stiffness k (moment/unit length) and where translation of $\bar{\zeta}$ is opposed by springs exhibiting a distributed stiffness K (force/unit length). Eqs. (1)–(8) (or the appropriate other boundary conditions for the beam) represent the hydroelastic problem with a one-sided elastic partial cover of the liquid surface.

In addition we have to observe the preservation of the liquid volume, i.e. for the clamped–free case

$$\int_{z=-h}^{\bar{\zeta}}\int_{x=0}^{\alpha l}\mathrm{d}x\,\mathrm{d}z+\int_{z=-h}^{\zeta}\int_{x=\alpha l}^{l}\mathrm{d}x\,\mathrm{d}z=hl, \tag{12}$$

which yields

$$\int_{x=0}^{\alpha l}\bar{\zeta}\,\mathrm{d}x+\int_{x=\alpha l}^{l}\zeta\,\mathrm{d}x=0. \tag{13}$$

3. Method of solution

The solution of the Laplace equation (1) together with the solid wall conditions at the sidewalls (2) and container bottom (3) yields for the velocity potential the expression

$$\Phi(x,z,t)=\left\{\Phi_0+\sum_{n=1}^{\infty}A_n\cosh\left[\frac{n\pi}{l}(z+h)\right]\cos\left(\frac{n\pi x}{l}\right)\right\}\mathrm{e}^{\mathrm{i}\omega t}, \tag{14}$$

while the solution of the beam equation (5) is with the deflection $\bar{\zeta}(x,t)=\mathrm{e}^{\mathrm{i}\omega t}\bar{\zeta}(x)$ given by

$$\begin{aligned}\bar{\zeta}(x)=&A_0\cos\left(\beta\frac{x}{l}\right)+B_0\sin\left(\beta\frac{x}{l}\right)+C_0\cosh\left(\beta\frac{x}{l}\right)+D_0\sinh\left(\beta\frac{x}{l}\right)+\frac{\mathrm{i}\omega\varrho_L l^4}{EI\beta^4}\Phi_0\\&-\frac{\mathrm{i}\omega\varrho_L l^4}{EI}\sum_{n=1}^{\infty}A_n\frac{\cosh(n\pi h/l)}{n^4\pi^4-\beta^4}\cos\left(\frac{n\pi x}{l}\right),\end{aligned} \tag{15}$$

where

$$\beta^4=\frac{(\omega^2\mu-\varrho_L g)l^4}{EI}>0. \tag{16}$$

Depending whether the coupled frequency ω is large or small, i.e. $\beta^4>0$ or <0 the beam Eq. (5) requires two different approaches. It may be mentioned that for $\alpha=0$, i.e. a completely free liquid surface, the natural frequencies of the unobstructed rectangular container (Lorell, 1951) would be obtained, while for $\alpha=1$ the liquid surface would be completely covered with an elastic plate. If the liquid density ϱ_L vanishes, Eq. (5) represents just the beam equation with $\beta^4\to\bar{\beta}^4=\mu\omega^2/EI$ and exhibits solution (15) with $\varrho_L=0$.

3.1. The case $\beta^4>0$

We shall treat here the case of a one-sided elastic cover reaching from $x=0$ to αl ($\alpha<1$) and being clamped–free, i.e. exhibiting the boundary conditions (7) and (8). They yield with Eq. (15)

$$A_0+C_0+\frac{\omega^*}{\mu^*\beta^4}\Phi_0^*-\frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty}A_n^*\frac{\cosh(n\pi h/l)}{n^4\pi^4-\beta^4}=0, \tag{17}$$

$$B_0 + D_0 = 0, \tag{18}$$

$$A_0\cos(\alpha\beta) + B_0\sin(\alpha\beta) - C_0\cosh(\alpha\beta) - D_0\sinh(\alpha\beta) - \frac{\omega^*\pi^2}{\mu^*\beta^2}\sum_{n=1}^{\infty} n^2 A_n^* \frac{\cosh(n\pi h/l)}{n^4\pi^4 - \beta^4}\cos(n\pi\alpha) = 0 \tag{19}$$

and

$$A_0\sin(\alpha\beta) - B_0\cos(\alpha\beta) + C_0\sinh(\alpha\beta) + D_0\cosh(\alpha\beta) - \frac{\omega^*\pi^3}{\mu^*\beta^3}\sum_{n=1}^{\infty} n^3 A_n^* \frac{\cosh(n\pi h/l)}{n^4\pi^4 - \beta^4}\sin(n\pi\alpha) = 0, \tag{20}$$

where

$$\omega^* = \omega l^2\sqrt{\frac{\mu}{EI}},\quad \mu^* = \frac{\mu}{\varrho_L l},\quad g^* = \frac{gl^3\mu}{EI},\quad \Phi_0^* = \mathrm{i}l\sqrt{\frac{\mu}{EI}}\Phi_0$$

and

$$A_n^* = \mathrm{i}lA_n\sqrt{\frac{\mu}{EI}},\quad \beta^4 = \omega^{*2} - \frac{g^*}{\mu^*}.$$

From Eqs. (17)–(20) the constants A_0, B_0, C_0 and D_0 may be determined. They depend on Φ_0^* and A_n^*. From the condition of constant liquid volume (13) follows the equation

$$A_0\sin(\alpha\beta) - B_0[\cos(\alpha\beta) - 1] + C_0\sinh(\alpha\beta) + D_0[\cosh(\alpha\beta) - 1] + \frac{\alpha\beta\omega^*}{\mu^*\beta^4}\Phi_0^*$$
$$- \frac{\omega^*\beta}{\mu^*}\sum_{n=1}^{\infty}\bar{A}_n^*\frac{\sin(n\pi\alpha)}{n\pi(n^4\pi^4 - \beta^4)} + \frac{\beta}{\omega^*}\sum_{n=1}^{\infty}\bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\sin(n\pi\alpha) = 0 \tag{21}$$

from which together with the results of Eqs. (17)–(20) Φ_0^* may be obtained. The remaining Eq. (4) for the free surface condition and (6) for the compatibility condition have to be satisfied at $z = 0$ in their respective range in x. If there is as treated here, one elastic member of length αl clamped-in at the solid wall $x = 0$ the range of the elastic plate (beam) is $0 \leqslant x/l \leqslant \alpha$, and that of the free liquid surface is given by $\alpha < x/l \leqslant 1$. The deflection of the beam is with the above notations presented by

$$\bar{\zeta}(x) = A_0\cos\left(\beta\frac{x}{l}\right) + B_0\sin\left(\beta\frac{x}{l}\right) + C_0\cosh\left(\beta\frac{x}{l}\right) + D_0\sinh\left(\beta\frac{x}{l}\right) + \frac{\omega^*}{\mu^*\beta^4}\Phi_0^*$$
$$- \frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty} A_n^*\frac{\cosh(n\pi h/l)}{n^4\pi^4 - \beta^4}\cos\left(\frac{n\pi x}{l}\right), \tag{22}$$

while the velocity potential is

$$\Phi(x,z,t) = \frac{-\mathrm{i}}{l\sqrt{\mu/EI}}\left\{\Phi_0^* + \sum_{n=1}^{\infty} A_n^*\cosh\left[\frac{n\pi}{l}(z+h)\right]\cos\left(\frac{n\pi x}{l}\right)\right\}\mathrm{e}^{\mathrm{i}\omega t}. \tag{23}$$

The compatibility condition (6) is with $\bar{A}_n^* = A_n^*\cosh(n\pi h/l)$ given by

$$A_0\cos\left(\beta\frac{x}{l}\right) + B_0\sin\left(\beta\frac{x}{l}\right) + C_0\cosh\left(\beta\frac{x}{l}\right)$$
$$+ D_0\sinh\left(\beta\frac{x}{l}\right) + \frac{\omega^*}{\mu^*\beta^4}\Phi_0^* - \frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty}\bar{A}_n^*\frac{1}{n^4\pi^4 - \beta^4}\cos\left(\frac{n\pi x}{l}\right)$$
$$+ \frac{\pi}{\omega^*}\sum_{n=1}^{\infty} n\bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\cos\left(\frac{n\pi x}{l}\right) = 0 \tag{24}$$

and is valid in the range $0 \leqslant x/l \leqslant \alpha$, while the free surface condition (4) yields with the surface tension parameter $\sigma^* = \sigma l^2/(EI)$ the expression

$$\omega^{*2}\Phi_0^* - \sum_{n=1}^{\infty}\bar{A}_n^*\left\{n\pi(g^* + \sigma^*\mu^* n^2\pi^2)\tanh\left(\frac{n\pi h}{l}\right) - \omega^{*2}\right\}\cos\left(\frac{n\pi x}{l}\right) = 0 \tag{25}$$

in the range $\alpha < x/l \leqslant 1$. It may be mentioned that in both equations A_0, B_0, C_0, D_0 and Φ_0^* are given functions of $\bar{A}_n^*$. Since Eq. (24) and (25) cannot be satisfied exactly by analytical means, we choose to satisfy them at certain freely chosen points x. Satisfying the compatibility condition (24) at N_1 points x/l in the range $0 \leqslant x/l \leqslant \alpha$ and the free surface condition (25) at N_2 points in the range $\alpha \leqslant x/l \leqslant 1$, we obtain $N_1 + N_2$ equations in $\bar{A}_n^*$, $n = 1, 2, \ldots, (N_1 + N_2)$. These algebraic equations are obtained by introducing in Eq. (24) $x/l = \alpha n_1/N_1$ for $n_1 = 1, 2, \ldots, N_1$, yielding N_1 algebraic equations, and in Eq. (25) $x/l = [\alpha + (1-\alpha)n_2/(N_2+1)]$ for $n_2 = 1, 2, \ldots, N_2$, resulting in N_2 algebraic equations. The vanishing coefficient determinant of this homogeneous algebraic system is the natural frequency equation for the determination of the lower coupled natural frequencies $\omega^* = \omega l^2\sqrt{\mu/EI}$. The above infinite series are now finite sums running from $n = 1$ to $(N_1 + N_2)$.

3.2. The case $\beta^4 < 0$

Small coupled frequencies ω, i.e., the case $\beta^4 < 0$ or $\beta^4 = -\beta^{*4}$, $\beta^{*4} = (\varrho_L g - \omega^2\mu)l^4/EI = g^*/\mu^* - \omega^{*2}$, $g^*/\mu^* > \omega^{*2}$, requires a different solution of the beam equation. The beam equation is therefore given by the expression

$$\bar{\zeta}^{\mathrm{IV}}(\xi) + \beta^{*4}\bar{\zeta}(\xi) = -\varrho_L\omega\frac{l^3}{\sqrt{\mu EI}}\left\{\Phi_0^* + \sum_{n=1}^{\infty}\bar{A}_n^*\cos(n\pi\xi)\right\}, \tag{26}$$

where

$$\xi = \frac{x}{l}, \quad \bar{A}_n^* = A_n^*\cosh\left(\frac{n\pi h}{l}\right) = \mathrm{i}lA_n\sqrt{\frac{\mu}{EI}}\cosh\left(\frac{n\pi h}{l}\right).$$

Its solution is

$$\begin{aligned}\bar{\zeta}(\xi) = {}& \mathrm{e}^{(\beta^*/\sqrt{2})\,\xi}\left[A_0^*\cos\left(\frac{\beta^*}{\sqrt{2}}\xi\right) + B_0^*\sin\left(\frac{\beta^*}{\sqrt{2}}\xi\right)\right] + \mathrm{e}^{-(\beta^*/\sqrt{2})\xi}\left[C_0^*\cos\left(\frac{\beta^*}{\sqrt{2}}\xi\right) + D_0^*\sin\left(\frac{\beta^*}{\sqrt{2}}\xi\right)\right] \\ & - \frac{\omega^*}{\mu^*\beta^{*4}}\Phi_0^* - \frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty}\frac{\bar{A}_n^*}{n^4\pi^4 + \beta^{*4}}\cos(n\pi\xi).\end{aligned} \tag{27}$$

In the case of the clamped–free beam the following relations are with the boundary conditions of the beam, i.e. the Eqs. (7) and (8) given by

$$A_0^* + C_0^* - \frac{\omega^*}{\mu^*\beta^{*4}}\Phi_0^* - \frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty}\frac{\bar{A}_n^*}{n^4\pi^4 + \beta^{*4}} = 0, \tag{28}$$

$$A_0^* + B_0^* - C_0^* + D_0^* = 0, \tag{29}$$

$$\begin{aligned}& \mathrm{e}^{\alpha\beta^*/\sqrt{2}}\left[B_0^*\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - A_0^*\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right] + \mathrm{e}^{-\alpha\beta^*/\sqrt{2}}\left[C_0^*\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - D_0^*\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right] \\ & \quad = -\frac{\omega^*\pi^2}{\mu^*\beta^{*2}}\sum_{n=1}^{\infty}\frac{n^2\bar{A}_n^*\cos(n\pi\alpha)}{n^4\pi^4 + \beta^{*4}},\end{aligned} \tag{30}$$

$$\begin{aligned}& \mathrm{e}^{\alpha\beta^*/\sqrt{2}}\left\{B_0^*\left[\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - \sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right] - A_0^*\left[\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) + \cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right]\right\} \\ & \quad - \mathrm{e}^{-\alpha\beta^*/\sqrt{2}}\left\{C_0^*\left[\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - \cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right]\right. \\ & \quad \left. - D_0^*\left[\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) + \sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right]\right\} = \frac{\omega^*\pi^3\sqrt{2}}{\mu^*\beta^{*3}}\sum_{n=1}^{\infty}\frac{n^3\bar{A}_n^*\sin(n\pi\alpha)}{n^4\pi^4 + \beta^{*4}},\end{aligned} \tag{31}$$

while from the condition of constant liquid volume we obtain

$$\begin{aligned}& \mathrm{e}^{\alpha\beta^*/\sqrt{2}}\left\{A_0^*\left[\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) + \sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right] + B_0^*\left[\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - \cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right]\right\} \\ & \quad - \mathrm{e}^{-\alpha\beta^*/\sqrt{2}}\left\{C_0^*\left[\cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) - \sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right] + D_0^*\left[\sin\left(\frac{\alpha\beta^*}{\sqrt{2}}\right) + \cos\left(\frac{\alpha\beta^*}{\sqrt{2}}\right)\right]\right\} \\ & \quad - A_0 + B_0 + C_0 + D_0 - \frac{\sqrt{2}\alpha\omega^*}{\mu^*\beta^{*3}}\Phi_0^* - \frac{\sqrt{2}\omega^*\beta^*}{\mu^*}\sum_{n=1}^{\infty}\bar{A}_n^*\frac{\sin(n\pi\alpha)}{n\pi(n^4\pi^4 + \beta^{*4})} \\ & \quad + \frac{\sqrt{2}\beta^*}{\omega^*}\sum_{n=1}^{\infty}\bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\sin(n\pi\alpha) = 0.\end{aligned} \tag{32}$$

The integration constants $A_0^*,\ B_0^*,\ C_0^*,\ D_0^*$ and Φ_0^* may be obtained as functions of the constants $\bar{A}_n^*$ from the inhomogeneous algebraic equations (28)–(31) and (32). Conditions (6) and (4), i.e., the compatibility condition and the free surface condition are then

$$\mathrm{e}^{(\beta^*/\sqrt{2})\,\xi}\left[A_0^*\cos\left(\frac{\beta^*}{\sqrt{2}}\xi\right)+B_0^*\sin\left(\frac{\beta^*}{\sqrt{2}}\xi\right)\right]+\mathrm{e}^{-(\beta^*/\sqrt{2})\xi}\left[C_0^*\cos\left(\frac{\beta^*}{\sqrt{2}}\xi\right)+D_0^*\sin\left(\frac{\beta^*}{\sqrt{2}}\xi\right)\right]$$
$$-\frac{\omega^*}{\mu^*\beta^{*4}}\Phi_0^*-\frac{\omega^*}{\mu^*}\sum_{n=1}^{\infty}\frac{\bar{A}_n^*}{n^4\pi^4+\beta^{*4}}\cos(n\pi\xi)+\frac{\pi}{\omega^*}\sum_{n=1}^{\infty}n\bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\cos(n\pi\xi)=0 \tag{33}$$

and

$$\omega^{*2}\Phi_0^*-\sum_{n=1}^{\infty}\bar{A}_n^*\left[n\pi g^*\left(1+\frac{n^2\pi^2}{Bo}\right)\tanh\left(\frac{n\pi h}{l}\right)-\omega^{*2}\right]\cos(n\pi\xi)=0. \tag{34}$$

Satisfying the compatibility condition (33) at N_1 points x/l in the range $0\leqslant x/l\leqslant\alpha$ and the free surface condition (34) at N_2 points in the range $\alpha\leqslant x/l\leqslant 1$, we obtain (N_1+N_2) equations in $\bar{A}_n^*,\ n=1,2,\ldots,(N_1+N_2)$. These homogeneous algebraic equations are obtained by introducing in Eq. (33) $x/l=\alpha n_1/N_1$ for $n_1=1,2,\ldots,N_1$, yielding N_1 algebraic equations, and in Eq. (34) $x/l=[\alpha+(1-\alpha)n_2/(N_2+1)]$ for $n_2=1,2,\ldots,N_2$, resulting in N_2 algebraic equations. The vanishing coefficient determinant is the natural frequency equation for the determination of the lower coupled natural frequencies $\omega^*=\omega l^2\sqrt{\mu/EI}$. The above infinite series are now finite sums running from $n=1$ to (N_1+N_2).

4. Forced container excitations

If the container is harmonically excited in x-direction by $x_0\mathrm{e}^{\mathrm{i}\Omega t}$, where x_0 is the excitation amplitude and Ω the forcing frequency, the liquid in the container as well as the elastic partial cover will respond according to the magnitude of the amplitude x_0 and the forcing frequency Ω. The same is true for rotational (pitching) excitation $\theta_0\mathrm{e}^{\mathrm{i}\Omega t}$. In these cases we have to determine the magnification functions (response functions) of the free surface displacement $\zeta(x,t)$ and the plate displacement $\bar{\zeta}(x,t)$.

4.1. Harmonic translation

For harmonic translational excitation in x-direction we have to solve the Laplace equation (1) with the container bottom condition (3) and the sidewall conditions

$$\frac{\partial\Phi}{\partial x}=\mathrm{i}\Omega x_0\mathrm{e}^{\mathrm{i}\Omega t}\quad\text{at } x=0\quad\text{and}\quad x=l \tag{35}$$

and the free surface condition (4) together with the plate (beam) Eq. (5) with its boundary conditions (7) and (8) as well as the compatibility condition (6). Extracting the rigid body motion of the container by the transformation

$$\Phi(x,z,t)=\mathrm{e}^{\mathrm{i}\Omega t}[\mathrm{i}\Omega x_0x+\Psi(x,z)] \tag{36}$$

yields homogeneous wall boundary conditions for Ψ at $x=0,\ x=l$ and $z=-h$ and the Laplace equation $\Delta\Psi=0$. The free liquid surface condition (4) renders the expression

$$g\frac{\partial\Psi}{\partial z}-\frac{\sigma}{\varrho_L}\frac{\partial^3\Psi}{\partial x^2\partial z}-\Omega^2\Psi=\mathrm{i}\Omega^3x_0x\quad\text{at } z=0 \text{ in the range } \alpha l<x\leqslant l, \tag{37}$$

and the compatibility condition

$$\frac{\partial\bar{\zeta}}{\partial t}=\mathrm{e}^{\mathrm{i}\Omega t}\frac{\partial\Psi}{\partial z}\quad\text{at } z=0 \text{ in the range } 0\leqslant x\leqslant\alpha l. \tag{38}$$

The plate equation (beam equation) (5) with $\bar{\zeta}(x,t)=\mathrm{e}^{\mathrm{i}\Omega t}\bar{\zeta}(x)$ yields then

$$EI\frac{d^4\bar{\zeta}}{dx^4}-(\mu\Omega^2-\varrho_Lg)\bar{\zeta}=-\mathrm{i}\Omega\varrho_L\Psi_{|z=0}+\Omega^2\varrho_Lx_0x \tag{39}$$

and has to be solved with the corresponding boundary conditions. With

$$\Psi(x,z)=\Psi_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right)\cosh\left[\frac{n\pi}{l}(z+h)\right] \tag{40}$$

Eq. (39) yields the solution

$$\bar{\zeta}(x) = A_0 \cos\left(\bar{\beta}\frac{x}{l}\right) + B_0 \sin\left(\bar{\beta}\frac{x}{l}\right) + C_0 \cosh\left(\bar{\beta}\frac{x}{l}\right) + D_0 \sinh\left(\bar{\beta}\frac{x}{l}\right) + \frac{\Omega^*}{\mu^*\bar{\beta}^4}\Psi_0^* - \frac{\Omega^*}{\mu^*}\sum_{n=1}^{\infty} \bar{A}_n^* \frac{1}{n^4\pi^4 - \bar{\beta}^4}\cos\left(\frac{n\pi x}{l}\right) - \frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^4 l}x, \tag{41}$$

where $\bar{\beta}^4 = (\mu\Omega^2 - \varrho_L g)l^4/EI = \Omega^{*2} - g^*/\mu^*$, $\Omega^* = \Omega l^2\sqrt{\mu/EI}$ and $\Psi_0^* = \mathrm{i}l\sqrt{\mu/EI}\,\Psi_0$.

With the boundary condition of the beam we obtain

$$A_0 + C_0 + \frac{\Omega^*}{\mu^*\bar{\beta}^4}\Psi_0^* - \frac{\Omega^*}{\mu^*}\sum_{n=1}^{\infty} \bar{A}_n^* \frac{1}{n^4\pi^4 - \bar{\beta}^4} = 0, \tag{42}$$

$$B_0 + D_0 - \frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^5} = 0, \tag{43}$$

$$A_0\cos(\alpha\bar{\beta}) + B_0\sin(\alpha\bar{\beta}) - C_0\cosh(\alpha\bar{\beta}) - D_0\sinh(\alpha\bar{\beta}) = \frac{\Omega^*\pi^2}{\mu^*\bar{\beta}^2}\sum_{n=1}^{\infty} n^2\bar{A}_n^*\frac{\cos(n\pi\alpha)}{n^4\pi^4 - \bar{\beta}^4}, \tag{44}$$

$$A_0\sin(\alpha\bar{\beta}) - B_0\cos(\alpha\bar{\beta}) + C_0\sinh(\alpha\bar{\beta}) + D_0\cosh(\alpha\bar{\beta}) = \frac{\Omega^*\pi^3}{\mu^*\bar{\beta}^3}\sum_{n=1}^{\infty} n^3\bar{A}_n^*\frac{\sin(n\pi\alpha)}{n^4\pi^4 - \bar{\beta}^4}. \tag{45}$$

The velocity potential is (with Eq. (40))

$$\Phi(x,z,t) = \frac{\mathrm{i}e^{\mathrm{i}\Omega t}}{l\sqrt{\frac{\mu}{EI}}}\left\{\Omega^*\frac{x_0}{l}x - \Psi_0^* - \sum_{n=1}^{\infty} A_n^*\cos\left(\frac{n\pi x}{l}\right)\cosh\left[\frac{n\pi}{l}(z+h)\right]\right\} \tag{46}$$

and the compatibility condition yields

$$A_0 \cos\left(\bar{\beta}\frac{x}{l}\right) + B_0 \sin\left(\bar{\beta}\frac{x}{l}\right) + C_0 \cosh\left(\bar{\beta}\frac{x}{l}\right) + D_0 \sinh\left(\bar{\beta}\frac{x}{l}\right) + \frac{\Omega^*}{\mu^*\bar{\beta}^4}\Psi_0^* - \frac{\Omega^*}{\mu^*}\sum_{n=1}^{\infty} \bar{A}_n^* \frac{1}{n^4\pi^4 - \bar{\beta}^4}\cos\left(\frac{n\pi x}{l}\right) - \frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^4 l}x + \frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^5}\sinh\left(\bar{\beta}\frac{x}{l}\right) + \frac{\pi}{\Omega^*}\sum_{n=1}^{\infty} n\bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\cos\left(\frac{n\pi x}{l}\right) = 0, \tag{47}$$

which is valid in the range $0 \leqslant x/l \leqslant \alpha$, while the free surface condition results in the expression

$$\Omega^{*2}\Psi_0^* - \sum_{n=1}^{\infty} \bar{A}_n^*\left[n\pi(g^* + \sigma^*\mu^*n^2\pi^2)\tanh\left(\frac{n\pi h}{l}\right) - \Omega^{*2}\right]\cos\left(\frac{n\pi x}{l}\right) = \Omega^{*3}x_0\frac{x}{l} \tag{48}$$

and is to be applied in the range $\alpha \leqslant x/l \leqslant 1$. The condition of constant liquid volume (13) yields

$$A_0\sin(\alpha\bar{\beta}) - B_0[\cos(\alpha\bar{\beta}) - 1] + C_0\sinh(\alpha\bar{\beta}) + D_0[\cosh(\alpha\bar{\beta}) - 1] + \frac{\alpha\Omega^*}{\mu^*\beta^3}\Psi_0^* - \frac{\Omega^*\bar{\beta}}{\mu^*}\sum_{n=1}^{\infty} \bar{A}_n^*\frac{\sin(n\pi\alpha)}{n\pi(n^4\pi^4 - \beta^4)} - \frac{\alpha^2\Omega^{*2}x_0}{2\mu^*\bar{\beta}^3} + \frac{\bar{\beta}}{\Omega^*}\sum_{n=1}^{\infty} \bar{A}_n^*\tanh\left(\frac{n\pi h}{l}\right)\sin(n\pi\alpha) = 0 \tag{49}$$

Eqs. (47) and (48) are a set of inhomogeneous algebraic equations, that may be satisfied at freely chosen points x/l yielding for $n = 1, 2, \ldots, (N_1 + N_2)$ the response values $A_n^*(\Omega^*)$. We may notice that we deal with the same Eqs. (24) and (25), where the right-hand side zeros are replaced by

$$\frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^4 l}x - \frac{\Omega^{*2}x_0}{\mu^*\bar{\beta}^5}\sinh\left(\bar{\beta}\frac{x}{l}\right) \quad \text{and} \quad \Omega^{*3}x_0\frac{x}{l},$$

respectively, and ω^* by Ω^*. For the range $\bar{\beta}^4 < 0$ the procedure is similar as for the free oscillation case. We have just to observe the analysis of Section 3.2.

4.2. Rotational excitation

If the container is pitching with $\theta_0 e^{i\Omega t}$ about the y-axis through the center of mass of the liquid the problem may be solved by solving the Laplace equation (1) with the rigid wall conditions

$$\frac{\partial \Phi}{\partial x} = i\Omega\theta_0 z e^{i\Omega t} \quad \text{at } x = \pm\frac{l}{2} \tag{50}$$

and

$$\frac{\partial \Phi}{\partial z} = -i\Omega\theta_0 x e^{i\Omega t} \quad \text{at } z = -\frac{h}{2} \tag{51}$$

as well as the previous equation at the free surface $z = 0$. With the transformation

$$\Phi(x, z, t) = e^{i\Omega t}[i\Omega\theta_0 xz + \Psi(x, z)] \tag{52}$$

we obtain

$$\Delta\Psi = 0 \quad \text{and} \quad \frac{\partial \Psi}{\partial x} = 0 \quad \text{at } x = \pm\frac{l}{2} \tag{53}$$

and

$$\frac{\partial \Psi}{\partial z} = -2i\Omega\theta_0 x \quad \text{at } z = -\frac{h}{2}. \tag{54}$$

The free liquid surface condition (4) yields

$$g\frac{\partial \Psi}{\partial z} - \frac{\sigma}{\varrho_L}\frac{\partial^3 \Psi}{\partial z \partial x^2} - \Omega^2\Psi = i\Omega\theta_0\left(\frac{\Omega^2 h}{2} - g\right)x \quad \text{at } z = \frac{h}{2} \text{ in the range } \alpha - \frac{1}{2} < \frac{x}{l} \leqslant \frac{1}{2}. \tag{55}$$

The compatibility condition (6) is then given by

$$\frac{\partial \bar{\zeta}}{\partial t} = \left[\frac{\partial \Psi}{\partial z} + i\Omega\theta_0 x\right]e^{i\Omega t} \quad \text{at } z = \frac{h}{2} \text{ in the range } \frac{1}{2} < \frac{x}{l} \leqslant \alpha - \frac{1}{2}. \tag{56}$$

The plate equation is given by

$$\frac{d^4\bar{\zeta}}{dx^4} - \frac{\bar{\beta}^4}{l^4}\bar{\zeta} = -\frac{i\Omega\varrho_L}{EI}\Psi_{|z=h/2} + \frac{\Omega^2\varrho_L\theta_0 h}{2EI}x \tag{57}$$

and has to be solved with the boundary conditions. The procedure is similar as that performed above and is omitted here.

5. Numerical evaluations and conclusions

Some of the above analytical results have been evaluated numerically and are presented in graphical form. First of all, it should be mentioned that the magnitude of N_1 and N_2, i.e., the number of points at which the mixed surface conditions are satisfied, varies according to the magnitude of the beam, that partly covers the free liquid surface. For $\alpha = 0.2$, i.e., a length of the beam of $0.2l$ and thus covering one fifth of the liquid surface, N_1 and N_2 were chosen to be $N_1 = 3$ and $N_2 = 24$. They were sufficiently large to yield good approximate results for the lower natural frequencies. If the beam covers half of the liquid surface, i.e., if $\alpha = 0.5$, the numerical procedure employed $N_1 = 20$ and $N_2 = 25$ and presented satisfactory results, while for a further increase of the beam to $\alpha = 0.8$, covering an amount of eighty percent of the free liquid surface area a further increase of the number of points used on the beam was $N_1 = 40$, while the small free liquid surface area required only $N_2 = 13$, in order to yield acceptable results for the natural frequencies as well as for the coupled frequencies of the liquid–structure system. The magnitude of N_1 and N_2 was chosen, such that the results of a further increase did not considerably change them.

In Fig. 2 we represent for a Bond number $\mathrm{Bo} = \varrho_L g l^2/\sigma = 1.3438 \times 10^5$, a mass density ratio $\mu^* = \mu/\varrho_L l = 0.4$ and a gravity parameter of $g^* = g\mu l^3/EI = 1.7938 \times 10^{-3}$ the first natural frequency for $\alpha = 0,\ 0.2,\ 0.5$ and 0.8. The liquid was chosen to be water with the density $\varrho_L = 10^3\ \mathrm{kg/m^3}$ and the beam of steel exhibited Young's modulus of elasticity $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, while the density is $\varrho_C = 8 \times 10^3\ \mathrm{kg/m^3}$. The surface tension of water is given by $\sigma = 7.3 \times 10^{-2}\ \mathrm{N/m}$. The thickness of the beam $d = 5$ cm is for this figure rather large and presents to the system a nearly rigid beam structure. The natural liquid frequency $\omega/\sqrt{g/l}$ is presented as a function of the liquid height ratio h/l ($0 \leqslant h/l \leqslant 1$). The dashed line shows the natural liquid frequency for $\alpha = 0$, i.e., for a free liquid surface not

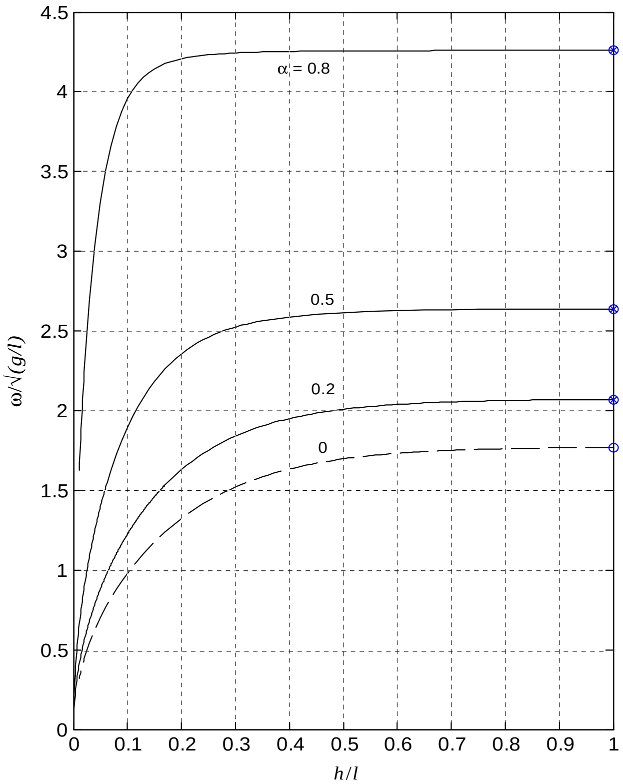

Fig. 2. Fundamental natural frequency of partly covered free surface (rigid obstruction) for Bo $= 134380$, $g^* = 0.0017938$, $\mu^* = 0.4$.

obstructed by a beam. The natural frequency increases with increasing liquid height ratio h/l and reaches soon its limit magnitude

$$\frac{\omega_{nL}}{\sqrt{g/l}} = \sqrt{n\pi\left(1+\frac{n^2\pi^2}{Bo}\right)\tanh\left(n\pi\frac{h}{l}\right)}$$

for $n = 1$. For higher modes $n > 1$ the asymptotic value is practically reached for much smaller h/l-values. The fundamental ($n = 1$) natural frequency for an obstructed liquid in a rectangular container is shown for $\alpha = 0.2$, 0.5 and 0.8 and exhibits with increased covering of the liquid surface increased natural frequencies. The here presented results of a nearly rigid beam structure agrees with the results of Bauer and Eidel (2000) obtained previously. The larger the area of obstruction, the less influence is noticed from the liquid height ratio h/l (see the case $\alpha = 0.8$). Only for small h/l the natural frequencies decrease rapidly.

In Figs. 3a–c we represent the coupled frequencies as a function of the thickness d of the beam. In addition we indicate the uncoupled frequencies ($n = 1, 2, 3$) for the liquid with a rigid beam and the uncoupled frequencies ($n = 1, 2, 3$) for the beam without any interaction with the liquid, i.e., $\varrho_L = 0$. The uncoupled natural frequencies, presented as dash–dotted lines (-·-·) are obtained for a clamped–free beam case, for which $\lambda_1 = 1.8751$, $\lambda_2 = 4.6941$, $\lambda_3 = 7.8548$. With

$$\omega_{nB} = \frac{\lambda_n^2}{\alpha^2 l^2}\sqrt{\frac{EI}{\mu}} = \frac{\lambda_n^2 d}{\alpha^{3/2} l^{3/2}}\sqrt{\frac{E}{12\varrho_C}}$$

we notice that the structural frequency is proportional to the thickness d of the beam and indirectly proportional to the 3/2-power of the beam length ($\alpha^{-3/2} = 11.19, 2.83, 1.4$ for $\alpha = 0.2, 0.5, 0.8$, respectively). For the numerical evaluation the length l of the container was chosen to be one meter ($l = 1$ m). Fig. 3a represents the coupled frequencies for a short beam of $\alpha = 0.2$. The dashed lines represent the natural frequencies for the liquid without a rigid structural cover. At $d = 0$ the frequency is only that of the liquid ($\alpha = 0$) at the height ratio $h/l = 1$. It is indicated by the mark $\circ$. The natural frequencies with an elastic liquid cover increase with increasing beam thickness d and approach for large

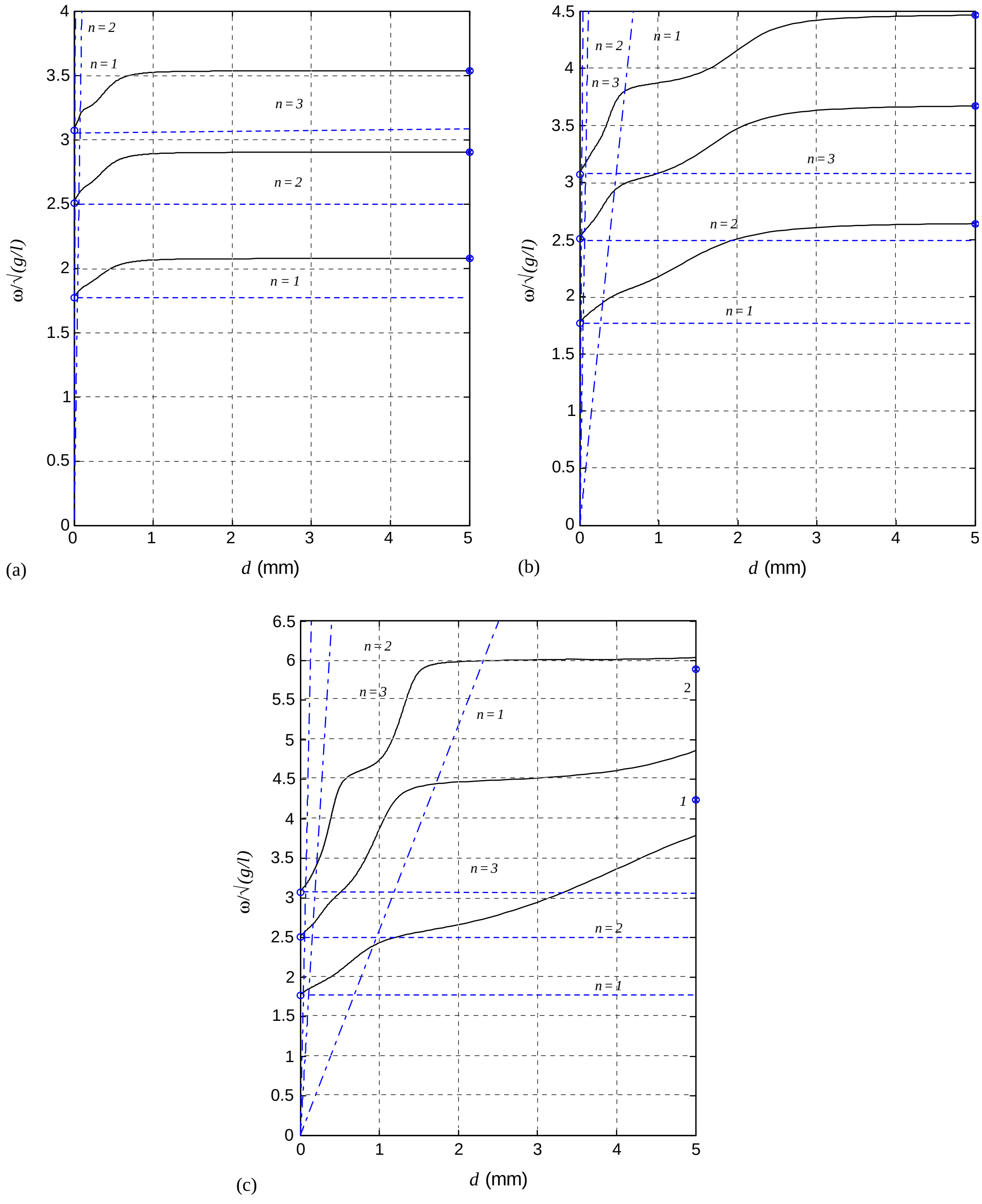

Fig. 3. (a) Coupled hydroelastic frequencies of partly covered free surface (elastic obstruction) for $\alpha = 0.2$, $\varrho_L = 1000\ \mathrm{kg/m^3}$, $\sigma = 0.073\ \mathrm{N/m}$, $l = 1$ m, $h/l = 1$, $\varrho_C = 8000\ \mathrm{kg/m^3}$, $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, $N_1 = 3$, $N_2 = 24$. (b) Coupled hydroelastic frequencies of partly covered free surface (elastic obstruction) for $\alpha = 0.5$, $\varrho_L = 1000\ \mathrm{kg/m^3}$, $\sigma = 0.073\ \mathrm{N/m}$, $l = 1$ m, $h/l = 1$, $\varrho_C = 8000\ \mathrm{kg/m^3}$, $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, $N_1 = 3$, $N_2 = 24$. (c) Coupled hydroelastic frequencies of partly covered free surface (elastic obstruction) for $\alpha = 0.8$, $\varrho_L = 1000\ \mathrm{kg/m^3}$, $\sigma = 0.073\ \mathrm{N/m}$, $l = 1$ m, $h/l = 1$, $\varrho_C = 8000\ \mathrm{kg/m^3}$, $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, $N_1 = 3$, $N_2 = 24$.

d-values the case of the rigid beam structure, i.e., the magnitude for $\alpha = 0.2$ at $h/l = 1$ of Fig. 2. It is represented by $\otimes$-signs. We notice that for a short beam ($\alpha = 0.2$) the elasticity of the structure reduces the natural frequencies in a range, which increases with the increase of the modal form. The higher the mode the larger the frequency range with the change of the thickness of the beam. It should, however, be mentioned that mainly the fundamental modes are of engineering interest for dynamic studies. This is due to the fact that the fundamental sloshing and beam mode exhibit the largest generalized masses interacting on each other. Since the coupled frequency $\omega/\sqrt{g/l}$ is smaller than $\varrho_L l/\varrho_C d$ the values of Fig. 3a have all been obtained with the above results of case 3.2, i.e., $\beta^4 < 0$ ($\beta^{*4} > 0$). Fig. 3b shows the coupled frequencies for $\alpha = 0.5$, i.e., the case, where half the liquid surface is covered by the beam. Here we notice a stronger interaction of structure and liquid in the lower thickness range. The results in this case have also been obtained by applying the solution of the case 3.2, i.e., $\beta^{*4} > 0$. In addition the frequency ranges become increasingly larger with the change of the thickness of the beam. It may be noticed that the beam thickness of 5 mm ($d = 5$ mm) yields for short beams ($\alpha = 0.2,\ 0.5$) natural frequencies, which are very close to those of the rigid beam structure ($\otimes$). If eighty percent of the free liquid surface is covered by the beam structure, the results are shown in Fig. 3c. Again we notice that the coupled frequencies increase with increasing thickness d of the beam and approach, however, at a considerable larger thickness of the beam (not presented here) finally the rigid beam case ($\otimes$). Since the liquid frequency for a rigid obstruction increases considerably with increasing α, i.e., increasing beam length, the change of the coupled frequency exhibits a large magnitude as the thickness of the beam increases. We notice that the coupled liquid frequencies are always located between the liquid frequency without obstruction ($\alpha = 0$) and that of the liquid frequency with a rigid partial surface cover. The coupled liquid frequencies are always smaller than those of the liquid in a container with rigid obstructions. It may be mentioned that the result of the third coupled frequency ($n = 3$) in Fig. 3c is obtained by applying both cases 3.1 and 3.2 as indicated in the figure. With decreasing thickness d the application of the case 3.1 shall require higher coupled frequencies (according to $\omega/\sqrt{g/l} > \varrho_L l/\varrho_C d$). The coupled structure-frequencies have not been presented, since their frequency ranges are of larger magnitude and outside the range of this presentations. The fluctuating magnitude of the natural frequency in the lower thickness range is due to the interaction of the elastic structure and exhibits with the increase of the beam length larger frequency changes (see Figs. 3a–c).

6. Conclusions

The above results lead us to the following conclusions:

(i) the covering of a part of the free liquid surface increases the natural frequencies of the liquid;
(ii) if the surface obstruction is rigid the natural frequencies of the liquid increase with increasing cover area;
(iii) if the surface cover exhibits elasticity the coupled liquid–structure frequencies show in comparison with the rigid cover decreased magnitude, the decrease of which becomes larger for increasing cover area (α), decreasing beam thickness d and increasing mode.

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